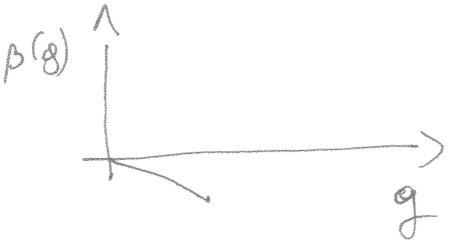
1.1 Asymptotic freedom (AF) is a property of the Yang-Mills theory and it refers to the fact that the theory becomes free, i.e the coupling g goes to zero asymptotically in the ultraviolet (UV), i.e. as the energy (length) goes to infinity (zero).

Hence, AF manifests itself in the fact that the beta function, $\beta(g) = -\dfrac{dg}{d\log z}$, vanishes at zero coupling and it is negative (at least) in an infinitesimal neighborhood of the origin.



Close to $g=0$ the one-loop perturbative beta function establishes exactly the presence or absence of AF.

Given $\beta(g) = -\beta_0\, g^3 + \cdots$

AF is realized for $\beta_0 > 0$.

1.2 AF is lost when β_0 changes sign.

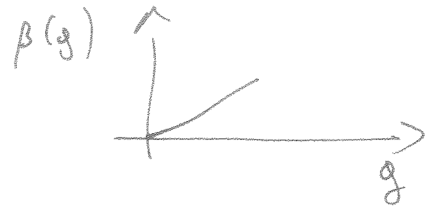
This can happen by coupling the pure YM gauge theory to a sufficient amount of, e.g., Dirac fermions.

For QCD-like theories with N_f flavors in the fundamental representation AF is lost for $\beta_0 \leq 0$, i.e.

$$\frac{11}{3} N - \frac{2}{3} N_f \leq 0$$

$$N_f \geq \frac{11}{2} N$$

Indeed, for $N_f > \frac{11}{2} N$ the beta function close to $g=0$ is positive

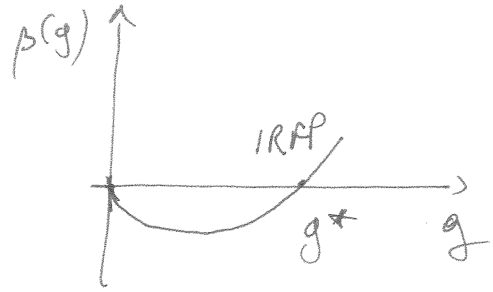


The theory is now IR free.

1.3 Based on the 2-loop beta function

$$\beta(g) = -\beta_0 g^3 - \beta_1 g^5 + \dots$$

a nontrivial IR fixed point (IRFP) occurs in the AF region, i.e. $\beta_0 > 0$, when $\beta_1 < 0$



The IRFP coupling is solution to the equation $\beta(g)=0$
At two loops:

$$-\beta_0 g^3 - \beta_1 g^5 = 0$$

and the nontrivial IRFP occurs at $g^{*2} = -\frac{\beta_0}{\beta_1}$

The lowest N_f for which an IRFP occurs according to the 2-loop beta function is solution to $\beta_1 = 0$:

$$\frac{34}{3} N^2 - \frac{13}{3} N N_f + \frac{N_f}{N} = 0$$

$$N_f = \frac{34}{13} \frac{N}{1 - \frac{3}{13N^2}}$$

Thus, according to the 2-loop beta function a conformal window (CW) exists for

$$\frac{34}{13}\frac{N}{\left(1-\frac{3}{13N^2}\right)} \le N_f < \frac{11}{2}N$$

Clearly, we cannot trust the perturbative prediction away from a neighborhood of $g=0$ and, therefore, we cannot trust the prediction ~~for~~ above for the lower edge of the CW.

Extra: Comment on the Veneziano limit and the exact prediction of the existence of an IRFP close to $N_f = \frac{11}{2}N$

1.4
$$-\frac{dg}{d\log z} = -\beta_0 g^3$$

$$\int_{g(\mu^{-1})}^{g(x)} \frac{dg}{g^3} = \beta_0 \int_{\mu^{-1}}^{x} d\log z$$

$$-\frac{1}{2}\left(\frac{1}{g^2(x)} - \frac{1}{g^2(\mu^{-1})}\right) = \beta_0 \log(x\mu)$$

$$\frac{1}{g^2(x)} = \frac{1}{g^2(\mu^{-1})} - 2\beta_0 \log(x\mu).$$

$$g^2(x) = \frac{g^2(\mu^{-1})}{1-g^2(\mu^{-1})\beta_0 \log(x^2\mu^2)}$$

$$\approx g^2(\mu^{-1})\left(1+g^2(\mu^{-1})\beta_0\log(x^2\mu^2) + O(g^4)\right)$$

2. $[\mathcal{L}] = 4$ for $d = 4$

- $[H]$ from kinetic term $\partial_\mu H^\dagger \partial^\mu H$

$$2[\partial_\mu] + 2[H] = 4$$

$$[H] = 1 \qquad \left([H] = \frac{d-2}{2} \right)$$

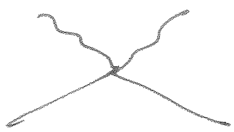
- $[W_\mu^a]$ from kinetic term $-\frac{1}{4} W_{\mu\nu}^a W^{\mu\nu a}$

$$2[\partial_\mu] + 2[W_\mu^a] = 4$$

$$[W_\mu^a] = 1$$

Analogously for B_μ : $[B_\mu] = 1$

Then $[g]$, $[g']$, $[\lambda]_4$ follow from dimensional analysis of the corresponding vertices



$$2[g] + 2[W_\mu^a] + 2[H] = 4$$

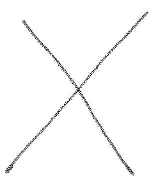
$$[g] = 0$$

equivalently from the triple vertex

$$[g] + [\partial_\mu] + [W_\mu^a] + 2[H] = 4$$

$$[g] = 0$$

Analogously for B_μ : $[g'] = 0$



$$4[H] + [\lambda] = 4$$

$$[\lambda] = 0$$

For $[g]=[g']=[\lambda]=0$ D does not depend on any of the number of vertices $V_{3,4,\lambda}$:

$$D = 4 - E_H - E_G$$

This implies that there is a finite number of 1PI primitively UV divergent amplitudes (not diagrams) those that have $D \geq 0$, i.e., $E_H + E_G \leq 4$.

Therefore, the theory is renormalizable.

3. In the massless limit $m_q = 0$ we denote by A_u the amplitude for any of the up quarks u, c, t. Analogously we denote by A_d the one for any of the down quarks d, s, b.

$$A_u = \frac{ie}{4 \sin\theta_W \cos\theta_W} \bar{u}_z(p_1) \gamma^\mu (a_u - \gamma_5) v_s(p_2) \varepsilon_\mu^\alpha(k)$$

$$A_u^+ = \frac{-ie}{4 s_W c_W} \bar{v}_s(p_2) \gamma^\nu (a_u - \gamma_5) u_z(p_1) \varepsilon_\nu^\alpha(k)$$

$$X_u = \frac{1}{3} \frac{e^2}{16 s_W^2 c_W^2} \mathrm{Tr}\left[\not{p}_1 \gamma^\mu (a_u - \gamma_5) \not{p}_2 \gamma^\nu (a_u - \gamma_5)\right]\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M_Z^2}\right)$$

$$\mathrm{Tr}[\ldots] = \mathrm{Tr}\left[\not{p}_1 \gamma^\mu \not{p}_2 \gamma^\nu (1 + a_u^2 - 2 a_u \gamma_5)\right]$$

The term $\propto \gamma_5$ does not contribute since $\mathrm{Tr}[\gamma_5 \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta] \propto \varepsilon^{\mu\nu\alpha\beta}$ is totally antisymmetric and it is contracted with the symmetric tensor $-g_{\mu\nu} + \frac{k_\mu k_\nu}{M_Z^2}$

Thus:

$$X_u = \frac{1}{3}\frac{e^2}{s_w^2 c_w^2}\frac{1}{16}4\left(1+a_u^2\right)\left(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - g^{\mu\nu} p_1 p_2\right) \times$$
$$\times\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M_Z^2}\right)$$

$$= \frac{1}{12}\frac{e^2}{s_w^2 c_w^2}\left(1+a_u^2\right)\left(p_1 p_2 + \frac{2(kp_1)(kp_2)}{M_Z^2}\right)$$

Kinematics:

$$p_1 + p_2 = k \qquad k = (M_Z, \vec{0}) \qquad p_1 = (E, \vec{p}) \quad p_2 = (E, -\vec{p})$$

Massless limit: $E = |\vec{p}|$

$$(p_1+p_2)^2 = k^2$$

$$2p_1p_2 = k^2 \quad \Longrightarrow \quad p_1p_2 = \frac{M_Z^2}{2}$$

$$kp_1 = p_1^2 + p_1p_2 = p_1p_2 \quad \Longrightarrow \quad kp_1 = kp_2 = p_1p_2 = \frac{M_Z^2}{2}$$

$$kp_2 = p_2^2 + p_1p_2 = p_1p_2$$

$$X_u = \frac{e^2}{12 s_w^2 c_w^2}\left(1+a_u^2\right)M_Z^2$$

Analogously

$$X_d = \frac{e^2}{12 s_w^2 c_w^2}\left(1+a_d^2\right)M_Z^2$$

The total X, after summing over the 3 up quarks and 3 down quarks is:

$$X = 3X_u + 3X_d = \frac{e^2 M_Z^2}{4\sin^2\theta_W \cos^2\theta_W}\left(2 + a_u^2 + a_d^2\right)$$